Newton polygons

Setting: • (K, v) non-archimedean field

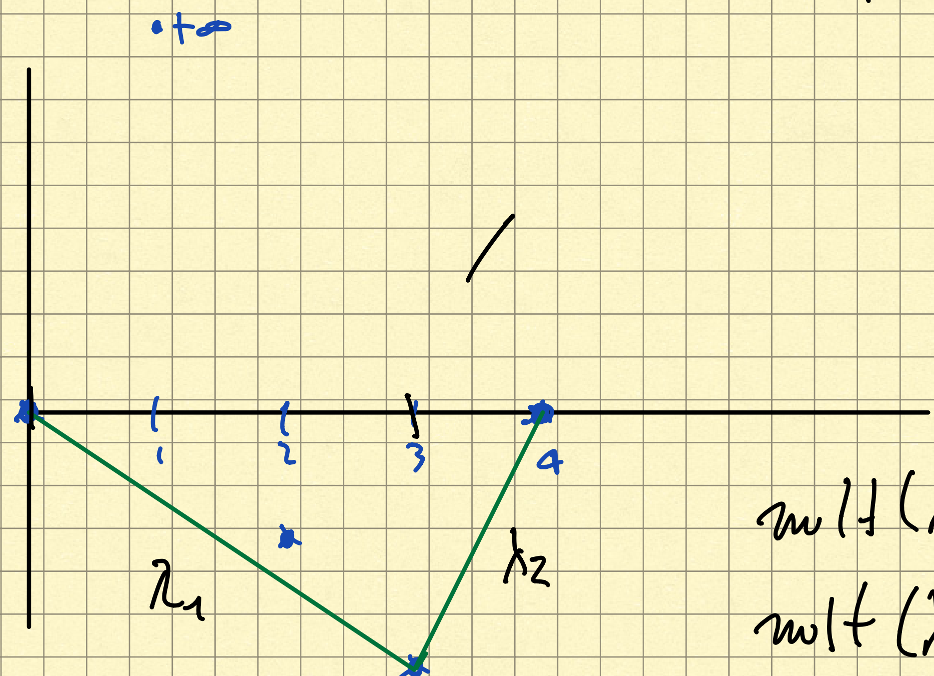
$$v: K \to \mathbb{R} \cup \{+\infty\}$$

• $f = \sum_{i=0}^{n} a_i t^i \in K[t]$.

Def: The Newton polygon of f, $NP(f)$, is the largest convex polygon below the set

$$\{ (i, v(a_i)) \in \mathbb{R}^2 : 0 \leq i \leq n \}$$

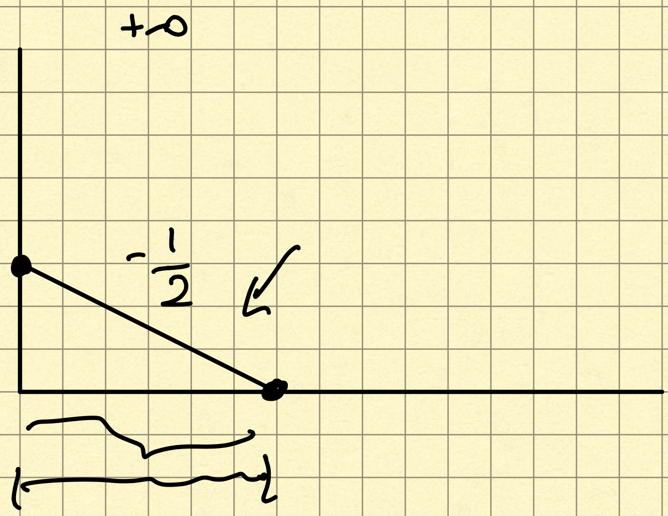
Example: $K = \mathbb{Q}_p$ $\quad f = 1 + \frac{t^2}{p} + \frac{t^3}{p^2} + t^4$



$\mathrm{mult}(\lambda_1) = 3$

$\mathrm{mult}(\lambda_2) = 1$

$f = t^2 - p$ if x is a root of f



$$v(x^2) = v(p) = 1$$
$$v(x^2) = 2v(x)$$
$$\Rightarrow \quad \underline{\underline{v(x) = \frac{1}{2}}}$$

Main Thm Newton polygons: letting $x_1, \dots, x_n \in \overline{K}^a$ be the roots of f, then

$$\{-v(x_i) : 1 \leq i \leq n\} = \text{slopes}(NP(f))$$

moreover if $\lambda \in \text{slopes}(NP(f))$ and $\text{mult}(\lambda) = m$, then there are $x_{i_1}, \dots, x_{i_m}$ s.t. $-v(x_{i_j}) = \lambda$.

- we want to see $NP(f)$ as a function $NP(f) : \mathbb{R} \to \overline{\mathbb{R}}$

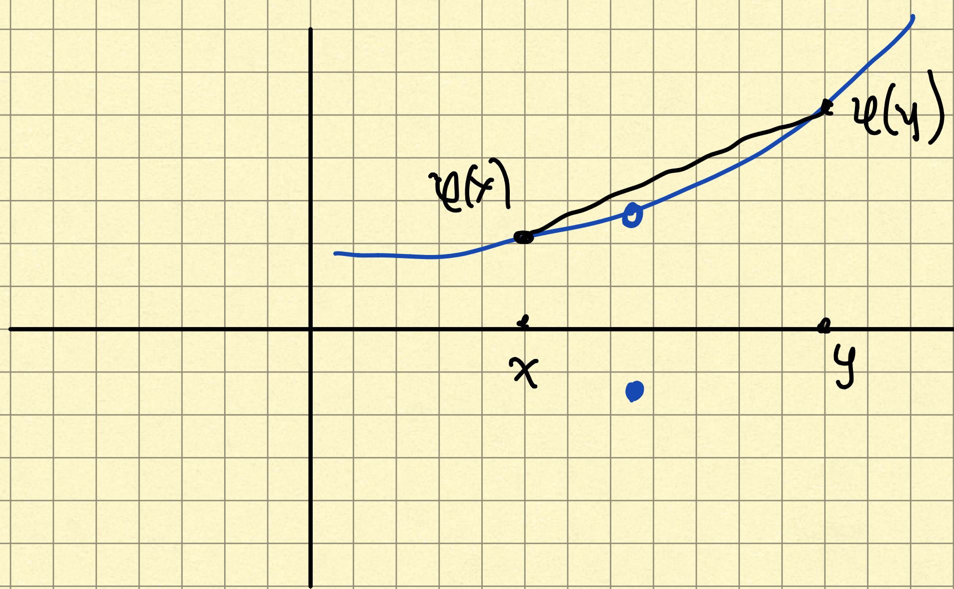
where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Let $\mathcal{F}$ be $\{ \varphi : \mathbb{R} \to \overline{\mathbb{R}} \}$.

Def: $\varphi \in \mathcal{F}$ is said to be convex (resp. concave) if for all $x, y \in \mathbb{R}$ and all $a, b \geq 0$ s.t. $a+b=1$

$$\varphi(ax+by) \leq a\varphi(x) + b\varphi(y)$$

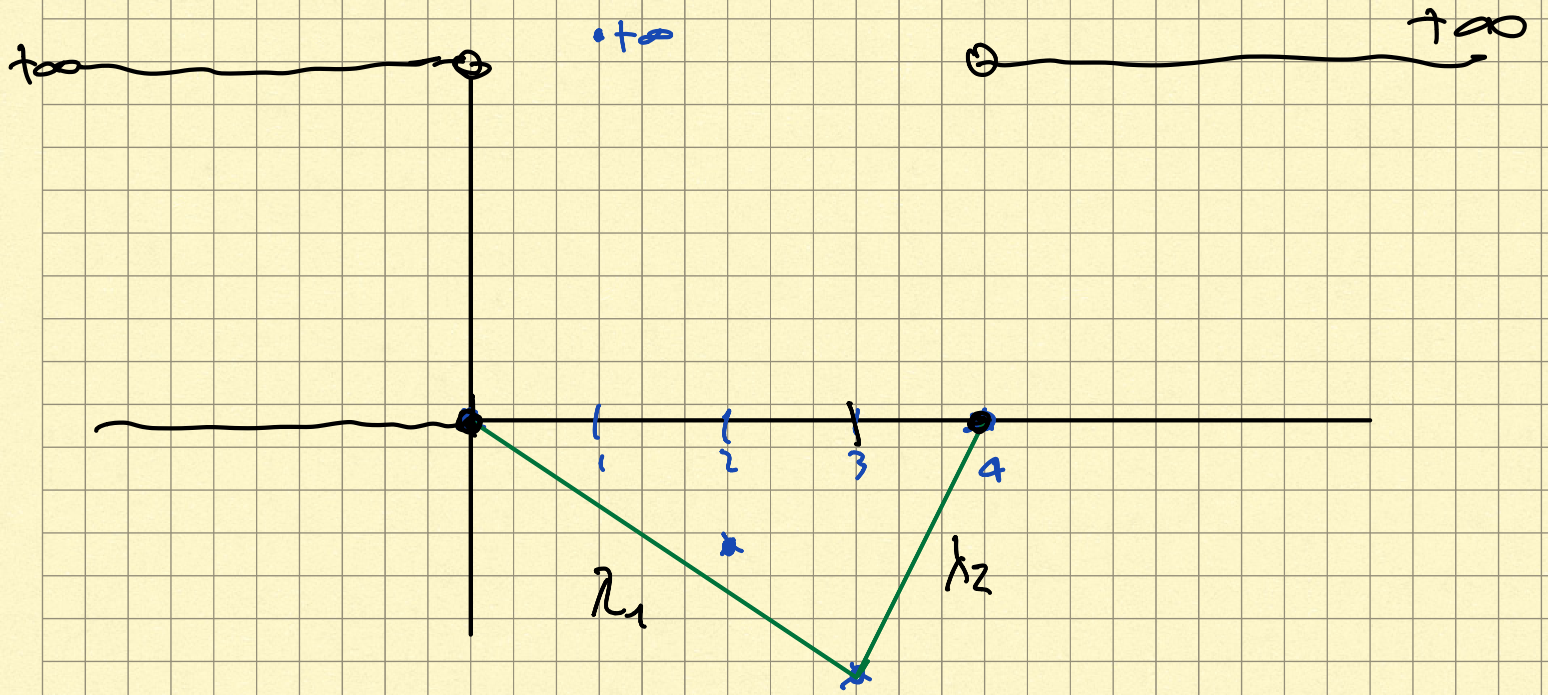
(resp. " $\geq$ ")



$NP(f)$ as a function is the largest convex function below the set

$$\{ (i, v(a_i)) \in \mathbb{R} \times \overline{\mathbb{R}} : i \in \mathbb{Z} \}$$

$a_i = 0$ if $i \in \{0, \ldots, n\}$



Corol. 6.16: For $f, g \in K[t]$

$$NP(f \cdot g) = NP(f) * NP(g)$$

where for $\varphi, \psi \in \mathcal{F}$ s.t. $-\infty \notin \mathrm{Im}(\varphi) \cup \mathrm{Im}(\psi)$

$$(\varphi * \psi)(x) = \inf_{a+b=x} \{ \varphi(a) + \psi(b) \}$$

Proof of the main thm of New. Poly:

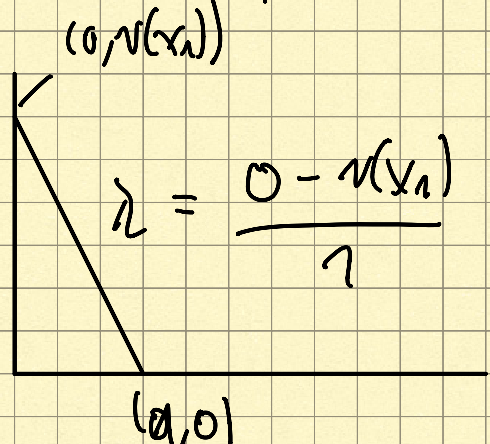
$$f = \alpha \cdot \prod_{i=1}^{n} (t - x_i)$$

we proceed by induction on n.

Reduction: WMA f is monic $(\alpha = 1)$

$$\{(i, v(\alpha \cdot a_i)) : i \in \mathbb{Z}\}$$
$$\{(i, v(\alpha) + v(a_i)) : i \in \mathbb{Z}\}$$

$n = 1$: $f = t - x_1$.

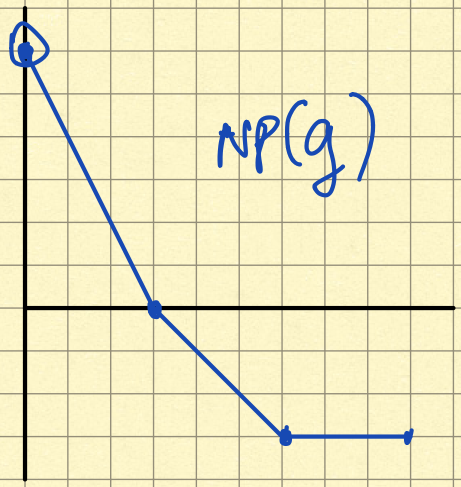


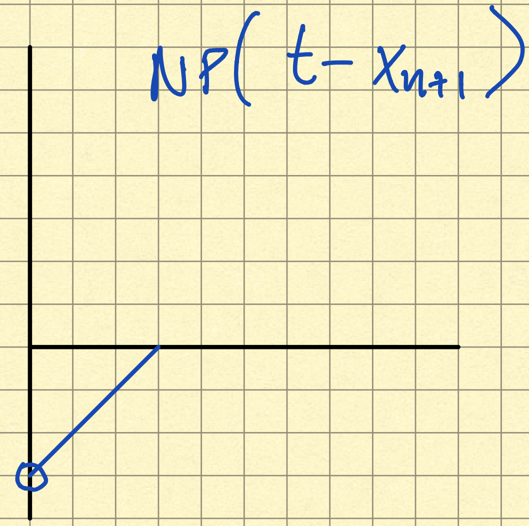
$$\lambda = \frac{0 - v(x_1)}{1} = -v(x_1)$$

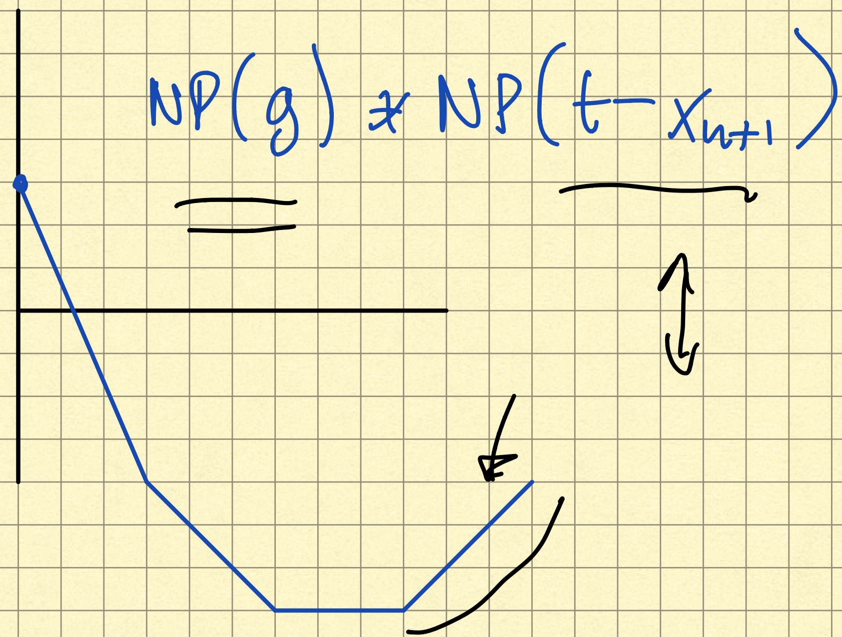
Suppose for n, $\deg(f) = n+1$

$$f = \underbrace{\left(\prod_{i=1}^{n} (t - x_i)\right)}_{g} (t - x_{n+1})$$

$$NP(f) = NP(g \cdot (t - x_{n+1})) \overset{\text{Lemma}}{=} NP(g) * NP(t - x_{n+1})$$







Legendre's transform:

$$\mathcal{L} : \mathcal{F} \longrightarrow \mathcal{F} \qquad \tilde{\mathcal{L}} : \mathcal{F} \longrightarrow \mathcal{F}$$

$$\varphi \longmapsto \mathcal{L}(\varphi)$$

$$\mathcal{L}(\varphi)(\lambda) = \inf_{x \in \mathbb{R}} \{ \varphi(x) + \lambda x \}$$

$$\tilde{\mathcal{L}}(\varphi)(\lambda) = \sup_{x \in \mathbb{R}} \{\varphi(x) - \lambda x\}$$

$$\tilde{\mathcal{L}}(\varphi) = -\mathcal{L}(-\varphi).$$

Lemma 6.15 : $\varphi, \psi \in \mathcal{F}$ and $-\infty \notin \mathrm{Im}(\varphi) \cup \mathrm{Im}(\psi)$ then

1) If φ, ψ are convex, so is $\varphi * \psi$.

2) $\mathcal{L}(\varphi * \psi) = \mathcal{L}(\varphi) + \mathcal{L}(\psi)$.

Remark : • If $\varphi \in \mathcal{F}$ convex (resp. concave) non-extendable then

$$\tilde{\mathcal{L}}(\mathcal{L}(\varphi)) = \varphi$$

(resp. $\mathcal{L}(\tilde{\mathcal{L}}(\varphi)) = \varphi$).

• Both $NP(f)$ and $NP(f) * NP(g)$ are convex non-extendable.

Lemma 6.13: $f, g \in K[t]$

$$\mathcal{L}(NP(f \cdot g)) = \mathcal{L}(NP(f)) + \mathcal{L}(NP(g)).$$

Proof of Cor 6.16 ($NP(fg) = NP(f) * NP(g)$)

By lemmas 6.13 and 6.15

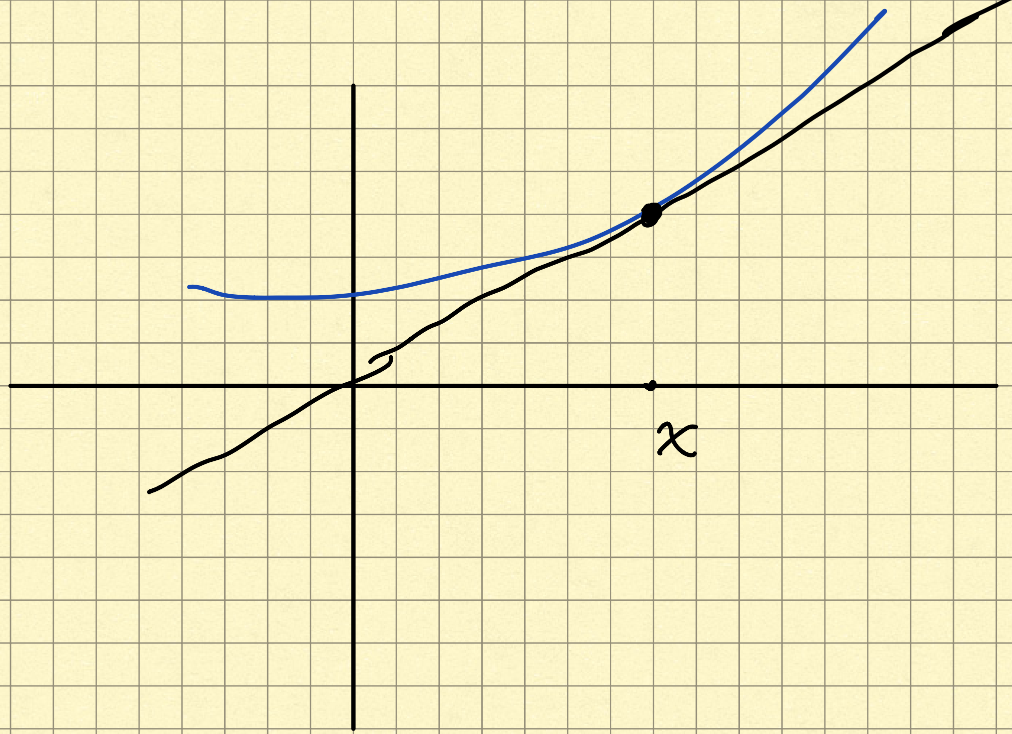
$$\mathcal{L}(NP(fg)) = \mathcal{L}(NP(f)) + \mathcal{L}(NP(g))$$
$$= \mathcal{L}(NP(f) * NP(g))$$

$\Rightarrow$ use the remark to obtain the result.

Def: For $\varphi \in \mathcal{F}$ and $x \in \mathbb{R}$, we say that φ admits a (resp. a capping line) supporting line of slope λ at x if $\varphi(x) \neq \pm\infty$ and

$$\forall y \in \mathbb{R} \quad \varphi(y) \geq \varphi(x) + \lambda(y-x)$$

$$\forall y \in \mathbb{R} \quad \varphi(y) \leq \varphi(x) + \lambda(y-x)$$

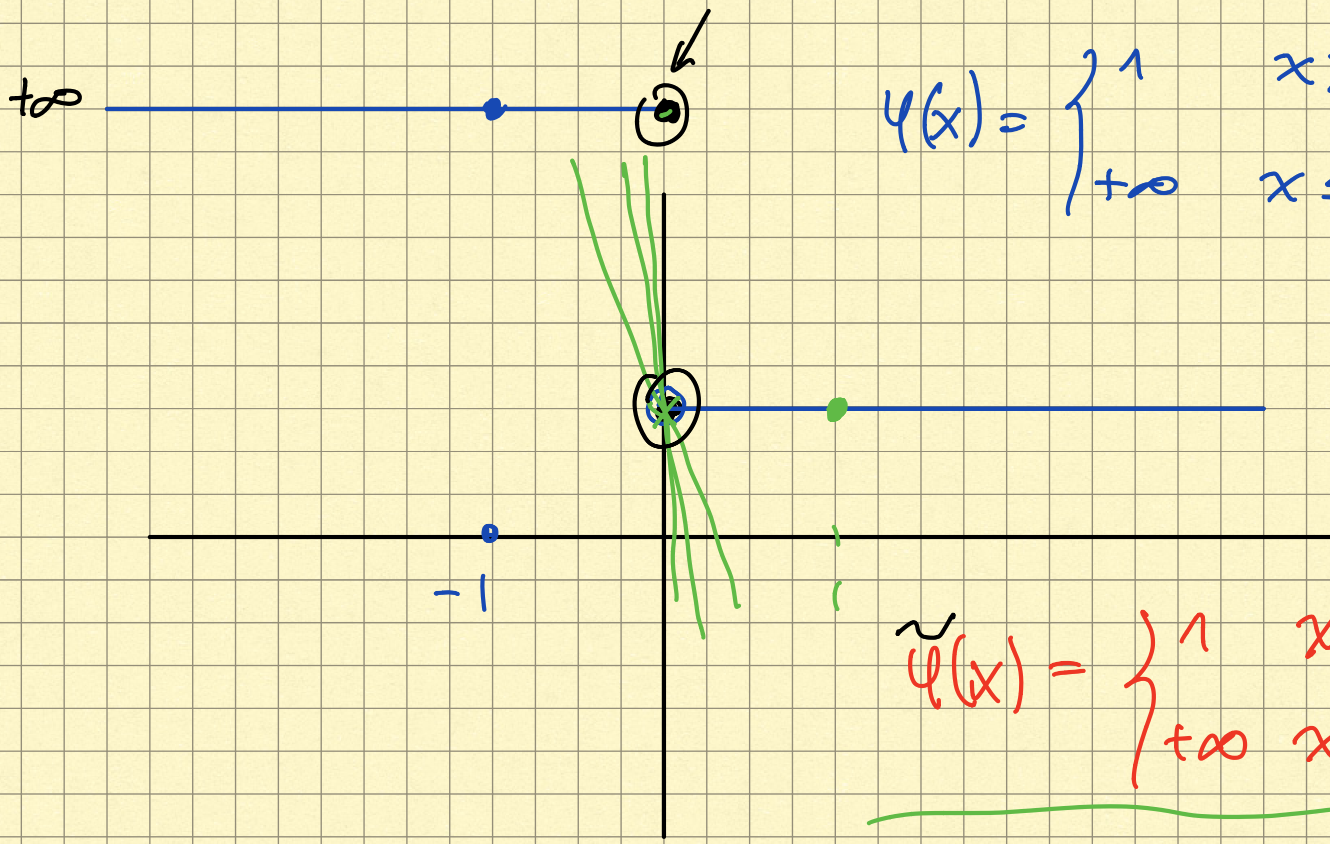


When $\varphi(x) = +\infty$ (resp. $-\infty$) we say

$c + \lambda(y-x)$ is a supp. line of slope λ at x if

$$\forall y \in \mathbb{R} \qquad \varphi(y) \geq \underline{\underline{c}} + \lambda(y-x).$$

Note: $\varphi \in \mathcal{F}$ is convex (resp. concave) if φ admits a supp. line (resp. a capping line) at every $x \in \mathbb{R}$.



$$\varphi(x) = \begin{cases} 1 & x > 0 \\ +\infty & x \le 0 . \end{cases}$$

$$\tilde{\varphi}(x) = \begin{cases} 1 & x \ge 0 \\ +\infty & x < 0 . \end{cases}$$

concave

Def: A convex $\varphi \in \Psi$ is non-extendable

infimum

if φ is the superum over all

supporting lines

capping line.

$$\mathcal{L}(\varphi) = \mathcal{L}(\tilde{\varphi})$$

Prop. 6.11: Let $\varphi, \psi \in \mathcal{F}$.

1) $\mathcal{L}(\varphi)$ is non-ext. concave, $\tilde{\mathcal{L}}(\varphi)$ is non-ext. convex.

2) if $\varphi \leq \psi$, $\mathcal{L}(\varphi) \leq \mathcal{L}(\psi)$
$\tilde{\mathcal{L}}(\varphi) \leq \tilde{\mathcal{L}}(\psi)$.

3) $\underline{\tilde{\mathcal{L}}\mathcal{L}(\varphi)} \leq \varphi$ and $\varphi \leq \mathcal{L}\tilde{\mathcal{L}}(\varphi)$.

4) If $\varphi \neq +\infty$, then if φ admits a supp line at x of slope λ, then $\mathcal{L}(\varphi)$ admits a capping line at $-\lambda$ of slope x.

5) $\mathcal{L}, \tilde{\mathcal{L}}$ define inverse bijections between convex non-ext. functions and concave non-ext. functions.

$$\mathcal{L}(NP(f))(r) = \inf_{i \in \mathbb{Z}} \{ v(a_i) + r \cdot i \} =: v_r(f)$$

$$v_r : K[t] \longrightarrow \mathbb{R} \cup \{+\infty\}. \quad \rightarrow \mathbb{A}_K^{1,an}.$$

a valuation

$r \in v(K^\times)$.

$$v_r(f \cdot g) = v_r(f) + v_r(g). \quad (*)$$

$\Downarrow$

implies Lemma 6.1.3.

Proof of $(*)$:

express $f = \prod_{i=1}^{n} (t - \alpha_i)$ we proceed by ind. on n.

(we only need to really do case 1).

$$v_r((t-\alpha_1) \cdot g) = v_r(t-\alpha_1) + v_r(g).$$

$$v_r(t\cdot g) = \underbrace{\underset{=}{r} + v_r(g)} = \inf\{v(b_i) + \underline{\underline{ri}}\}$$

$$g = \sum_{i=0}^{m} c_i t^i \qquad tg = \sum_{i=0}^{m} \underset{\uparrow b_i}{c_i} t^{i+1}$$

$$v_r(-\alpha_1 \cdot g) = \underbrace{v(\alpha_1) + v_r(g)}$$

$$v_r((t-\alpha_1)g) = v_r(tg - \alpha_1 g)$$

$$\geq \inf\{v_r(tg), v_r(-\alpha_1 g)\}$$

if $r \neq v(\alpha_1)$

$$v(tg - \alpha_1 g) = \inf\{v_r(tg), v(-\alpha_1 g)\}$$

$$= \inf\{r + v_r(g), v(a) + v_r(g)\}$$

$$= \inf\{r + v(\alpha_1)\} + v_r(g)$$

$$= v_r(t - \alpha_1) + v_r(g).$$

$r = v(\alpha_1)$.

look at the functions

$$r \longmapsto v_r(f) \qquad\qquad r \longmapsto v_r(f) + v_r(g)$$

these are continuous functions.

$$v_r((t-\alpha_1)g) = v_r(t-\alpha_1) + v_r(g)$$

for all $r \neq v(\alpha_1)$.

it must also hold at $r = v(\alpha_1)$
by the cont. of the above funct.

$f \in \mathcal{O}_K[[t]]$

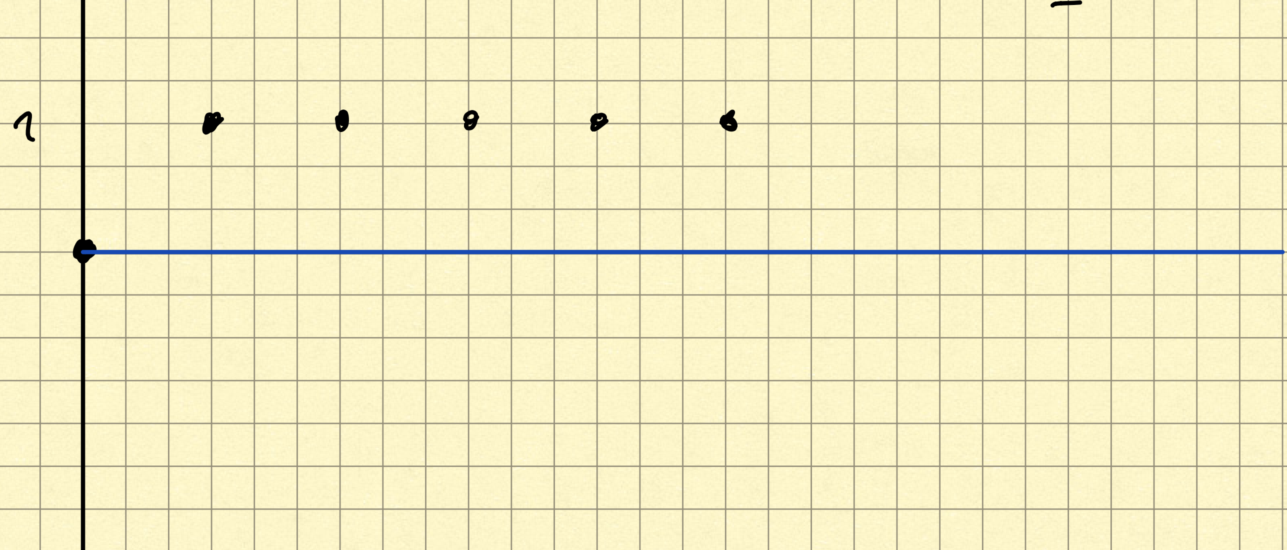
f defines a function on

$$\mathbb{D}_K = \{ x \in K : v(x) > 0 \}$$

Def: The Newton poly. of f, $NP(f)$, is the largest decreasing convex function below the set

$$\{ (i, v(a_i)) \in \mathbb{R} \times \overline{\mathbb{R}} : i \in \mathbb{Z} \}.$$

$$f = 1 + \underline{p}x + \underline{p}x^2 + \underline{p}x^3 + \dots$$



Thm 6.19 (Lazard) $f \in \mathcal{O}_K[\![t]\!]$ and $\lambda \neq 0$ is a slope of $NP(f)$, then there is $\alpha \in \overline{K}^a$ with $\underline{\underline{f(\alpha) = 0}}$ and $-v(\alpha) = \lambda$

$$f = (t-\alpha) \cdot g \qquad \text{for some } g$$

$$g \in \mathcal{O}_{\widehat{\overline{K}^a}}[\![t]\!]$$

$$1 + px + p^2x^2 + p^3x^3 + p^4x^4 + \dots$$